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The coarse shape groups

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ABSTRACT

The (pointed) coarse shape category Sh^* ($Sh^*_\star$), having (pointed) topological spaces as objects and having the (pointed) shape category as a subcategory, was recently constructed. Its isomorphisms classify (pointed) topological spaces strictly coarser than the (pointed) shape type classification. In this paper we introduce a new algebraic coarse shape invariant which is an invariant of shape and homotopy, as well. For every pointed space $(X,\star)$ and for every $k \in \mathbb{N}_0$, the coarse shape group $\check{\pi}^*_k(X,\star)$, having the standard shape group $\check{\pi}_k(X,\star)$ for its subgroup, is defined. Furthermore, a functor $\check{\pi}^*_k : Sh^*_\star \to Grp$ is constructed. The coarse shape and shape groups already differ on the class of polyhedra. An explicit formula for computing coarse shape groups of polyhedra is given. The coarse shape groups give us more information than the shape groups. Generally, $\check{\pi}_k(X,\star) = 0$ does not imply $\check{\pi}^*_k(X,\star) = 0$ (e.g. for solenoids), but from pro-$\pi_k(X,\star) = \mathbf{0}$ follows $\check{\pi}^*_k(X,\star) = 0$. Moreover, for pointed metric compacta $(X,\star)$, the n-shape connectedness is characterized by $\check{\pi}^*_k(X,\star) = 0$, for every $k \leqslant n$.

1. Introduction and preliminaries

Recently, N. Uglešić and the author in [2] have extended the shape theory by constructing a coarse shape category Sh^* whose objects are all topological spaces. Its isomorphisms classify topological spaces strictly coarser than the shape does. The shape category Sh can be considered a subcategory of Sh^*. The shape and coarse shape coincide on the class of spaces having homotopy type of polyhedra. In the same way the pointed coarse shape category (of pairs) $Sh^*_\star$ ($Sh^{*2}_\star$) having pointed topological spaces (pairs) for objects and having the pointed shape category (of pairs) $Sh_\star$ ($Sh^2_\star$) for its subcategory is constructed. The coarse shape, similar to shape, uses the technique of inverse systems, but they differ essentially. All fibres of a shape fibration over an arbitrary metric continuum, generally don't have the same shape type, but they are mutually coarse shape equivalent. The coarse shape preserves some important topological or shape invariants (see [4]) as connectedness, movability, strong movability, n-movability, shape dimension and stability. In the present paper we consider some new algebraic invariants of coarse shape and, consequently, of shape, along with applications to shape theory. We will introduce functors $\check{\pi}^*_n : Sh^*_\star \to Grp$ (Grp denotes the category of groups) and compare them with the standard shape functors $\check{\pi}_n : Sh_\star \to Grp$. The functor $\check{\pi}^*_n$ assigns to every pointed space $(X,\star)$ the n-th coarse shape group $\check{\pi}^*_n(X,\star)$ having the n-th shape group $\check{\pi}_n(X,\star)$ as its subgroup. Therefore, the coarse shape groups provide informations of pointed spaces which the

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shape groups are not able to give. It is proved that for pointed metric compacta $(X,\star)$ n-shape connectedness is equivalent to $\check{\pi}_k^*(X,\star)=0$, for every $0\leqslant k\leqslant n$.

We begin by recalling some of the main notions concerning the coarse shape category and the pro*-category (see [2]). Let $\mathcal{C}$ be a category and let $\boldsymbol{X}=(X_\lambda,p_{\lambda\lambda'},\Lambda)$ and $\boldsymbol{Y}=(Y_\mu,q_{\mu\mu'},M)$ be two inverse systems in $\mathcal{C}$. An *S^*-morphism of inverse systems*, $(f,f_\mu^n):\boldsymbol{X}\to\boldsymbol{Y}$, consists of an index function $f:M\to\Lambda$, and of a set of $\mathcal{C}$-morphisms $f_\mu^n:X_{f(\mu)}\to Y_\mu$, $n\in\mathbb{N}$, $\mu\in M$, such that, for every related pair $\mu\leqslant\mu'$ in M, there exists a $\lambda\in\Lambda$, $\lambda\geqslant f(\mu),f(\mu')$, and there exists an $n\in\mathbb{N}$ so that, for every $n'\geqslant n$,

$$f_\mu^{n'}p_{f(\mu)\lambda}=q_{\mu\mu'}f_{\mu'}^{n'}p_{f(\mu')\lambda}.$$

If $M=\Lambda$ and the index function is the identity 1_Λ and, for every pair $\lambda\leqslant\lambda'$, there exists an $n\in\mathbb{N}$ such that, for every $n'\geqslant n$, $f_\lambda^{n'}p_{\lambda\lambda'}=q_{\lambda\lambda'}f_{\lambda'}^{n'}$, then the S^*-morphism $(1_\Lambda,f_\lambda^n)$ is said to be *level*.

The *composition* of S^*-morphisms $(f,f_\mu^n):\boldsymbol{X}\to\boldsymbol{Y}$ and $(g,g_\nu^n):\boldsymbol{Y}\to\boldsymbol{Z}=(Z_\nu,r_{\nu\nu'},N)$ is an S^*-morphism $(h,h_\nu^n)=(g,g_\nu^n)(f,f_\mu^n):\boldsymbol{X}\to\boldsymbol{Z}$, where $h=fg$ and $h_\nu^n=g_\nu^nf_{g(\nu)}^n$. The *identity* S^*-morphism on $\boldsymbol{X}$ is an S^*-morphism $(1_\Lambda,1_{X_\lambda}^n):\boldsymbol{X}\to\boldsymbol{X}$, where 1_Λ is the identity function and $1_{X_\lambda}^n=1_{X_\lambda}$ are the identity morphisms in $\mathcal{C}$, for all $n\in\mathbb{N}$ and $\lambda\in\Lambda$.

An S^*-morphism $(f,f_\mu^n):\boldsymbol{X}\to\boldsymbol{Y}$ of inverse systems in $\mathcal{C}$ is said to be *equivalent to* an S^*-morphism $(f',f_\mu'^n):\boldsymbol{X}\to\boldsymbol{Y}$, denoted by $(f,f_\mu^n)\sim(f',f_\mu'^n)$, provided every $\mu\in M$ admits a $\lambda\in\Lambda$, $\lambda\geqslant f(\mu),f'(\mu)$, and an $n\in\mathbb{N}$, such that, for every $n'\geqslant n$,

$$f_\mu^{n'}p_{f(\mu)\lambda}=f_\mu'^{n'}p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation among S^*-morphisms of inverse systems in $\mathcal{C}$. The equivalence class $[(f,f_\mu^n)]$ of an S^*-morphism $(f,f_\mu^n):\boldsymbol{X}\to\boldsymbol{Y}$ is briefly denoted by $\boldsymbol{f}^*$.

The category pro*-$\mathcal{C}$ has as objects all inverse systems $\boldsymbol{X}$ in $\mathcal{C}$ and as morphisms all equivalence classes $\boldsymbol{f}^*=[(f,f_\mu^n)]$ of S^*-morphisms (f,f_μ^n). The composition in pro*-$\mathcal{C}$ is well defined by putting

$$\boldsymbol{g}^*\boldsymbol{f}^*=\boldsymbol{h}^*\equiv\big[\big(h,h_\nu^n\big)\big],$$

where $(h,h_\nu^n)=(g,g_\nu^n)(f,f_\mu^n)=(fg,g_\nu^nf_{g(\nu)}^n)$. For every inverse system $\boldsymbol{X}$ in $\mathcal{C}$, the identity morphism in pro*-$\mathcal{C}$ is $\boldsymbol{1}_{\boldsymbol{X}}^*=[(1_\Lambda,1_{X_\lambda})]$.

A functor $\underline{J}\equiv\underline{J}_{\mathcal{C}}:\text{pro-}\mathcal{C}\to\text{pro}^*\text{-}\mathcal{C}$ is defined as follows. It keeps objects fixed, i.e. $\underline{J}(\boldsymbol{X})=\boldsymbol{X}$, for every inverse system $\boldsymbol{X}$ in $\mathcal{C}$. If $\boldsymbol{f}\in\text{pro-}\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$ and if (f,f_μ) is any representative of $\boldsymbol{f}$, then a morphism $\underline{J}(\boldsymbol{f})=\boldsymbol{f}^*=[(f,f_\mu^n)]\in\text{pro}^*\text{-}\mathcal{C}(\boldsymbol{X},\boldsymbol{Y})$ is represented by the S^*-morphism (f,f_μ^n), where $f_\mu^n=f_\mu$ for all $\mu\in M$ and $n\in\mathbb{N}$. The morphism $\boldsymbol{f}^*$ is said to be *induced* by $\boldsymbol{f}$. Since the functor $\underline{J}$ is faithful, we may consider the category pro-$\mathcal{C}$ as a subcategory of pro*-$\mathcal{C}$. Thus, every morphism $\boldsymbol{f}$ in pro-$\mathcal{C}$ can be considered a morphism of the category pro*-$\mathcal{C}$, too.

Let $\mathcal{D}$ be a full and pro-reflective (i.e., dense) subcategory of $\mathcal{C}$ (see [5]). Let $\boldsymbol{p}:X\to\boldsymbol{X}$ and $\boldsymbol{p}':X\to\boldsymbol{X}'$ be $\mathcal{D}$-expansions of the same object X of $\mathcal{C}$, and let $\boldsymbol{q}:Y\to\boldsymbol{Y}$ and $\boldsymbol{q}':Y\to\boldsymbol{Y}'$ be $\mathcal{D}$-expansions of the same object Y of $\mathcal{C}$. Then there exist two natural (unique) isomorphisms $\boldsymbol{i}:\boldsymbol{X}\to\boldsymbol{X}'$ and $\boldsymbol{j}:\boldsymbol{Y}\to\boldsymbol{Y}'$ in pro-$\mathcal{D}$. Consequently, $\boldsymbol{i}^*\equiv\underline{J}(\boldsymbol{i}):\boldsymbol{X}\to\boldsymbol{X}'$ and $\boldsymbol{j}^*\equiv\underline{J}(\boldsymbol{j}):\boldsymbol{Y}\to\boldsymbol{Y}'$ are isomorphisms in pro*-$\mathcal{D}$. A morphism $\boldsymbol{f}^*:\boldsymbol{X}\to\boldsymbol{Y}$ is said to be *pro*-$\mathcal{D}$ equivalent to* a morphism $\boldsymbol{f}'^*:\boldsymbol{X}'\to\boldsymbol{Y}'$, denoted by $\boldsymbol{f}^*\sim\boldsymbol{f}'^*$, provided the following diagram in pro*-$\mathcal{D}$ commutes:

$$\begin{tikzcd}
\boldsymbol{X} \arrow[r, "\boldsymbol{i}^*"] \arrow[d, "\boldsymbol{f}^*"'] & \boldsymbol{X}' \arrow[d, "\boldsymbol{f}'^*"] \\
\boldsymbol{Y} \arrow[r, "\boldsymbol{j}^*"] & \boldsymbol{Y}'.
\end{tikzcd}$$

Hereby is defined an equivalence relation on the appropriate subclass of Mor(pro*-$\mathcal{D}$), such that $\boldsymbol{f}^*\sim\boldsymbol{f}'^*$ and $\boldsymbol{g}^*\sim\boldsymbol{g}'^*$ imply $\boldsymbol{g}^*\boldsymbol{f}^*\sim\boldsymbol{g}'^*\boldsymbol{f}'^*$ whenever it is defined. The equivalence class of $\boldsymbol{f}^*$ is denoted by $\langle\boldsymbol{f}^*\rangle$.

We define the (*abstract*) *coarse shape category* $Sh^*_{(\mathcal{C},\mathcal{D})}$ *for* $(\mathcal{C},\mathcal{D})$ as follows. The objects of $Sh^*_{(\mathcal{C},\mathcal{D})}$ are all the objects of $\mathcal{C}$. A morphism $F^*\in Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y)$ is a pro*-$\mathcal{D}$ equivalence class $\langle\boldsymbol{f}^*\rangle$ of a morphism $\boldsymbol{f}^*:\boldsymbol{X}\to\boldsymbol{Y}$, with respect to any choice of a pair of $\mathcal{D}$-expansions $\boldsymbol{p}:X\to\boldsymbol{X}$, $\boldsymbol{q}:Y\to\boldsymbol{Y}$. The *composition* of an $F^*:X\to Y$, $F^*=\langle\boldsymbol{f}^*\rangle$ and a $G^*:Y\to Z$, $G^*=\langle\boldsymbol{g}^*\rangle$, is defined by the representatives, i.e. $G^*F^*:X\to Z$, $G^*F^*=\langle\boldsymbol{g}^*\boldsymbol{f}^*\rangle$. The *identity coarse shape morphism* on an object X, $1_X^*:X\to X$, is the pro*-$\mathcal{D}$ equivalence class $\langle\boldsymbol{1}_{\boldsymbol{X}}^*\rangle$ of the identity morphism $\boldsymbol{1}_{\boldsymbol{X}}^*$ in pro*-$\mathcal{D}$. Since

$$Sh^*_{(\mathcal{C},\mathcal{D})}(X,Y)\approx\text{pro}^*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y}),$$

one may say that pro*-$\mathcal{D}$ is the realizing category for the coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ in the same way as pro-$\mathcal{D}$ is for the shape category $Sh_{(\mathcal{C},\mathcal{D})}$. If X and Y are isomorphic objects of $Sh^*_{(\mathcal{C},\mathcal{D})}$, then we say that they have the same *coarse shape type*, and we write $sh^*(X)=sh^*(Y)$. We denote by $J\equiv J_{(\mathcal{C},\mathcal{D})}:Sh_{(\mathcal{C},\mathcal{D})}\to Sh^*_{(\mathcal{C},\mathcal{D})}$ the faithful functor which keeps the objects fixed and whose morphisms are induced by the "inclusion" functor $\underline{J}:\text{pro-}\mathcal{D}\to\text{pro}^*\text{-}\mathcal{D}$. The functor $S^*\equiv S^*_{(\mathcal{C},\mathcal{D})}:\mathcal{C}\to Sh^*_{(\mathcal{C},\mathcal{D})}$, which factorizes as $S^*=J_{(\mathcal{C},\mathcal{D})}S$, where $S:\mathcal{C}\to Sh_{(\mathcal{C},\mathcal{D})}$ is the shape functor, we call *the coarse shape functor*.

As in the case of abstract shape, the most interesting example of the above construction is $\mathcal{C} = HTop$—the homotopy category of topological spaces and $\mathcal{D} = HPol$—the homotopy category of spaces having the homotopy type of polyhedra. In this case, one speaks about the (ordinary) *coarse shape category* $Sh^*_{(HTop,HPol)} \equiv Sh^*$ of topological spaces. The realizing category for Sh^* is the category pro*-$HPol$. On locally nice spaces (polyhedra, CW-complexes, ANR's) the coarse shape type classification coincides with the shape type classification and, consequently, with the homotopy type classification, but coarse shape differs from shape on the class of all topological spaces. Since the pointed homotopy category of polyhedra $HPol_\star$ is pro-reflective (i.e., dense) in the pointed homotopy category $HTop_\star$ (Theorem 1.4.7 in [5]), the *pointed coarse shape category* $Sh^*_{(HTop_\star,HPol_\star)} \equiv Sh^*_\star$ is well defined. For an inverse system $((X_\lambda,\star), p_{\lambda\lambda'}, \Lambda)$ in an $HPol_\star$-expansion $\boldsymbol{p} : (X,\star) \to ((X_\lambda,\star), p_{\lambda\lambda'}, \Lambda)$ of pointed space $(X,\star)$, we will use the abbreviation $(\boldsymbol{X},\star)$. One can also define the *pointed coarse shape category of pairs* $Sh^*_{(HTop^2_\star,HPol^2_\star)} \equiv Sh^{*2}_\star$ via the pointed homotopy category of pairs $HTop^2_\star$ and its pro-reflective subcategory $HPol^2_\star$ (the pointed homotopy category of polyhedral pairs).

2. Construction of the coarse shape groups

In this paper the homotopy class $[f]$ of a map f (briefly H-map), i.e. a morphism of the category $HTop$, $HTop_\star$ or $HTop^2_\star$, will be denoted by omitting the brackets. Recall that, for every pointed space $(X,\star)$ and for every $k \in \mathbb{N}_0$, elements of the k-dimensional homotopy group $\pi_k(X,\star)$ can be regarded as homotopy classes of maps $(S^k,\star) \to (X,\star)$, where $(S^k,\star)$ denotes the pointed k-dimensional sphere. We will use the additive notation for the group operation on $\pi_k(X,\star)$ for every $k \in \mathbb{N}$, although $\pi_k(X,\star)$ is abelian only for $k \geqslant 2$. The neutral element of $\pi_k(X,\star)$, i.e. the homotopy class of the map sending X to the base point $\star$, we will denote by $o : (S^k,\star) \to (X,\star)$. For the inverse of $a \in \pi_k(X,\star)$ we will use the notation $\bar{a}$. The trivial homotopy group $\pi_k(X,\star) = \{o\}$ we will denote by 0, same as any other trivial group.

Let $\check{\pi}^*_k(X,\star)$, for every $k \in \mathbb{N}_0$, denote the set of all coarse shape morphisms $A^* : (S^k,\star) \to (X,\star)$, i.e. the set $Sh^*_\star((S^k,\star),(X,\star))$. Let $A^*, B^* \in \check{\pi}^*_k(X,\star)$. Coarse shape morphisms A^* and B^* are represented by some morphisms $\boldsymbol{a}^* = [(a^n_\lambda)]$ and $\boldsymbol{b}^* = [(b^n_\lambda)] : (S^k,\star) \to (\boldsymbol{X},\star)$ in pro*-$HPol_\star$, respectively, where $\boldsymbol{p} : (X,\star) \to (\boldsymbol{X},\star) = ((X_\lambda,\star), p_{\lambda\lambda'}, \Lambda)$ is an $HPol_\star$-expansion of the pointed space $(X,\star)$. Notice, that the index functions of the S^*-morphisms (a^n_λ) and (b^n_λ) are omitted, since $(S^k,\star)$ is a rudimentary inverse system in $HPol_\star$. We will now define a binary operation on the set $\check{\pi}^*_k(X,\star)$, $k \in \mathbb{N}$, by putting

$$A^* + B^* = \langle \boldsymbol{a}^* \rangle + \langle \boldsymbol{b}^* \rangle = \langle \boldsymbol{a}^* + \boldsymbol{b}^* \rangle = \langle [(a^n_\lambda)] + [(b^n_\lambda)] \rangle = \langle [(a^n_\lambda + b^n_\lambda)] \rangle. \tag{1}$$

In order to verify that this operation is well defined first notice that $(a^n_\lambda + b^n_\lambda) : (S^k,\star) \to (\boldsymbol{X},\star)$ is an S^*-morphism. Indeed, for every pair $\lambda \leqslant \lambda'$, there exists an $n_1 \in \mathbb{N}$ such that $a^n_\lambda = p_{\lambda\lambda'} a^n_{\lambda'}$, for every $n \geqslant n_1$, and there exists an $n_2 \in \mathbb{N}$ such that $b^n_\lambda = p_{\lambda\lambda'} b^n_{\lambda'}$, for every $n \geqslant n_2$. Consequently, for every $n \geqslant \max\{n_1, n_2\}$ it holds

$$a^n_\lambda + b^n_\lambda = p_{\lambda\lambda'} a^n_{\lambda'} + p_{\lambda\lambda'} b^n_{\lambda'} = p_{\lambda\lambda'}\left(a^n_{\lambda'} + b^n_{\lambda'}\right),$$

which shows that $(a^n_\lambda + b^n_\lambda)$ is an S^*-morphism. Next we will prove that the sum operation $\boldsymbol{a}^* + \boldsymbol{b}^*$ of morphisms in pro*-$HPol_\star$ does not depend on their representatives (a^n_λ) and (b^n_λ). If $(a'^n_\lambda) \sim (a^n_\lambda)$ and $(b'^n_\lambda) \sim (b^n_\lambda)$, then for every λ, there exists an n_1 such that $a^n_\lambda = a'^n_\lambda$, for every $n \geqslant n_1$, and there exists an n_2 such that $b^n_\lambda = b'^n_\lambda$, for every $n \geqslant n_2$. Consequently, for every $n \geqslant \max\{n_1, n_2\}$, it holds

$$a^n_\lambda + b^n_\lambda = a'^n_\lambda + b'^n_\lambda,$$

which shows $(a^n_\lambda + b^n_\lambda) \sim (a'^n_\lambda + b'^n_\lambda)$. Finally, we need to prove that the sum operation in (1) of coarse shape morphisms A^* and B^* does not depend on their representatives $\boldsymbol{a}^*$ and $\boldsymbol{b}^*$ and the particular choice of an $HPol_\star$-expansion of the pointed space $(X,\star)$. Let $\boldsymbol{p}' : (X,\star) \to (\boldsymbol{X}',\star) = ((X'_\mu,\star), p'_{\mu\mu'}, M)$ be an $HPol_\star$-expansion of $(X,\star)$, and let $\boldsymbol{a}'^*$ and $\boldsymbol{b}'^* : (S^k,\star) \to (\boldsymbol{X}',\star)$ be representatives of A^* and B^* respectively. Therefore, $\boldsymbol{a}'^* \sim \boldsymbol{a}^*$ and $\boldsymbol{b}'^* \sim \boldsymbol{b}^*$. Hence, using the unique natural isomorphism $\boldsymbol{i}^* : \boldsymbol{X} \to \boldsymbol{X}'$ in pro*-$HPol_\star$ the following diagrams commute

$$\begin{tikzcd} (S^k,\star) \arrow[r, "\boldsymbol{a}^*"] \arrow[dr, "\boldsymbol{a}'^*"'] & (\boldsymbol{X},\star) \arrow[d, "\boldsymbol{i}^*"] \\ & (\boldsymbol{X}',\star) \end{tikzcd} \qquad \begin{tikzcd} (S^k,\star) \arrow[r, "\boldsymbol{b}'^*"] \arrow[dr, "\boldsymbol{b}^*"'] & (\boldsymbol{X},\star) \arrow[d, "\boldsymbol{i}^*"] \\ & (\boldsymbol{X}',\star). \end{tikzcd}$$

Let the S^*-morphism (i, i^n_μ) be the representative of $\boldsymbol{i}^*$. Then, we have

$$\begin{aligned} \boldsymbol{a}'^* + \boldsymbol{b}'^* &= \boldsymbol{i}^*\boldsymbol{a}^* + \boldsymbol{i}^*\boldsymbol{b}^* = \left[\left(i, i^n_\mu\right)\left(a^n_\lambda\right)\right] + \left[\left(i, i^n_\mu\right)\left(b^n_\lambda\right)\right] \\ &= \left[\left(i^n_\mu a^n_{i(\mu)}\right)\right] + \left[\left(i^n_\mu b^n_{i(\mu)}\right)\right] = \left[\left(i^n_\mu a^n_{i(\mu)} + i^n_\mu b^n_{i(\mu)}\right)\right] \\ &= \left[\left(i^n_\mu\left(a^n_{i(\mu)} + b^n_{i(\mu)}\right)\right)\right] = \left[\left(i, i^n_\mu\right)\left(a^n_\lambda + b^n_\lambda\right)\right] \\ &= \left[\left(i, i^n_\mu\right)\right]\left[\left(a^n_\lambda + b^n_\lambda\right)\right] = \boldsymbol{i}^*\left(\left[\left(a^n_\lambda\right)\right] + \left[\left(b^n_\lambda\right)\right]\right) = \boldsymbol{i}^*\left(\boldsymbol{a}^* + \boldsymbol{b}^*\right). \end{aligned}$$

It follows $\boldsymbol{a}'^* + \boldsymbol{b}'^* \sim \boldsymbol{a}^* + \boldsymbol{b}^*$, i.e. $\langle \boldsymbol{a}'^* + \boldsymbol{b}'^* \rangle = \langle \boldsymbol{a}^* + \boldsymbol{b}^* \rangle$, which completes the verification that the sum operation on $\check{\pi}_k^*(X,\star)$ is well defined by (1).

Theorem 1. *For every $k \in \mathbb{N}$, $\check{\pi}_k^*(X,\star)$ is a group with respect to the operation $+$. For $k \geqslant 2$, $\check{\pi}_k^*(X,\star)$ is an abelian group.*

Proof. Using associativity of the sum in $\pi_k(X_\lambda,\star)$, for every $\lambda \in \Lambda$, it is trivial to check that the sum in $\check{\pi}_k^*(X,\star)$ is associative. A coarse shape morphism $O^* : (S^k,\star) \to (X,\star)$ represented by $\boldsymbol{o}^* = [(o_\lambda^n)] : (S^k,\star) \to (\boldsymbol{X},\star)$, where $o_\lambda^n = o$, for every $n \in \mathbb{N}$ and $\lambda \in \Lambda$, is a two-sided neutral element in $\check{\pi}_k^*(X,\star)$. For an arbitrary $A^* = \langle \boldsymbol{a}^* \rangle \in \check{\pi}_k^*(X,\star)$, $\boldsymbol{a}^* = [(a_\lambda^n)] : (S^k,\star) \to (\boldsymbol{X},\star)$, we define a coarse shape morphism $\overline{A^*} = \langle \overline{\boldsymbol{a}^*} \rangle \in \check{\pi}_k^*(X,\star)$, represented by $\overline{\boldsymbol{a}^*} = [(\overline{a_\lambda^n})]$, where $\overline{a_\lambda^n} : (S^k,\star) \to (X_\lambda,\star)$ is the inverse for a_λ^n in $\pi_k(X_\lambda,\star)$, for every $n \in \mathbb{N}$ and $\lambda \in \Lambda$. First we need to prove that $(\overline{a_\lambda^n})$ is an S^*-morphism. Notice that for every pair $\lambda \leqslant \lambda'$ there exists $n \in \mathbb{N}$ such that $a_\lambda^{n'} = p_{\lambda\lambda'} a_{\lambda'}^{n'}$, for every $n' \geqslant n$. Hence, the homomorphism $\pi_k(p_{\lambda\lambda'})$ sends $a_{\lambda'}^{n'} \in \pi_k(X_{\lambda'},\star)$ to $a_\lambda^{n'} \in \pi_k(X_\lambda,\star)$, and consequently it sends the inverse $\overline{a_{\lambda'}^{n'}}$ of $a_{\lambda'}^{n'}$ to the inverse $\overline{a_\lambda^{n'}}$ of $a_\lambda^{n'}$, for every $n' \geqslant n$. Therefore, $\overline{a_\lambda^{n'}} = p_{\lambda\lambda'} \overline{a_{\lambda'}^{n'}}$, for every $n' \geqslant n$, which shows that $(\overline{a_\lambda^n})$ is an S^*-morphism. Now, it is trivial to prove that $\overline{A^*}$ is a two-sided inverse for A^* in $\check{\pi}_k^*(X,\star)$. Since for $k \geqslant 2$, $\pi_k(X_\lambda,\star)$ is an abelian group, for every $\lambda \in \Lambda$, one can easily verify that $A^* + B^* = B^* + A^*$ holds, for every $A^*, B^* \in \check{\pi}_k^*(X,\star)$, $k \geqslant 2$. $\square$

For every $k \in \mathbb{N}$ and for every coarse shape morphism $F^* : (X,\star) \to (Y,*)$ let us define a homomorphism

$$\check{\pi}_k^*(F^*) : \check{\pi}_k^*(X,\star) \to \check{\pi}_k^*(Y,*)$$

by the following rule

$$\check{\pi}_k^*(F^*)(A^*) = F^* A^*, \tag{2}$$

for every $A^* \in \check{\pi}_k^*(X,\star)$. In order to prove that $\check{\pi}_k^*(F^*)$ is a homomorphism suppose that A^* and $B^* \in \check{\pi}_k^*(X,\star)$ are represented by $\boldsymbol{a}^* = [(a_\lambda^n)]$ and $\boldsymbol{b}^* = [(b_\lambda^n)] : (S^k,\star) \to (\boldsymbol{X},\star)$, respectively, and F^* is represented by $\boldsymbol{f}^* = [(f, f_\mu^n)] : (\boldsymbol{X},\star) \to (\boldsymbol{Y},\star)$, where $\boldsymbol{p} : (X,\star) \to (\boldsymbol{X},\star) = ((X_\lambda,\star), p_{\lambda\lambda'}, \Lambda)$ is an $HPol_\star$-expansion of $(X,\star)$ and $\boldsymbol{q} : (Y,\star) \to (\boldsymbol{Y},\star) = ((Y_\mu,\star), q_{\mu\mu'}, M)$ is an $HPol_\star$-expansion of $(Y,\star)$. Now, we have

$$\begin{aligned}
&\check{\pi}_k^*(F^*)(A^*) + \check{\pi}_k^*(F^*)(B^*) \\
&\quad = F^*A^* + F^*B^* = \langle \boldsymbol{f}^*\boldsymbol{a}^* \rangle + \langle \boldsymbol{f}^*\boldsymbol{b}^* \rangle = \langle \boldsymbol{f}^*\boldsymbol{a}^* + \boldsymbol{f}^*\boldsymbol{b}^* \rangle \\
&\quad = \langle [(f, f_\mu^n)][(a_\lambda^n)] + [(f, f_\mu^n)][(b_\lambda^n)] \rangle = \langle [(f_\mu^n a_{f(\mu)}^n)] + [(f_\mu^n a_{f(\mu)}^n)] \rangle \\
&\quad = \langle [(f_\mu^n a_{f(\mu)}^n + f_\mu^n a_{f(\mu)}^n)] \rangle = \langle [(\pi_k(f_\mu^n)(a_{f(\mu)}^n) + \pi_k(f_\mu^n)(a_{f(\mu)}^n))] \rangle \\
&\quad = \langle [(\pi_k(f_\mu^n)(a_{f(\mu)}^n + a_{f(\mu)}^n))] \rangle = \langle [(f_\mu^n(a_{f(\mu)}^n + a_{f(\mu)}^n))] \rangle \\
&\quad = \langle [(f, f_\mu^n)(a_\lambda^n + a_\lambda^n)] \rangle = \langle [(f, f_\mu^n)][(a_\lambda^n + a_\lambda^n)] \rangle \\
&\quad = \langle \boldsymbol{f}^* \rangle \langle \boldsymbol{a}^* + \boldsymbol{b}^* \rangle = F^*(A^* + B^*) \\
&\quad = \check{\pi}_k^*(F^*)(A^* + B^*),
\end{aligned}$$

which implies that $\check{\pi}_k^*(F^*)$ is a homomorphism. Further, notice that for every pointed space $(X,\star)$, $\check{\pi}_k^*(1_X^*) : \check{\pi}_k^*(X,\star) \to \check{\pi}_k^*(X,\star)$ is the identity homomorphism, and for arbitrary coarse shape morphisms $F^* : (X,\star) \to (Y,\star)$ and $G^* : (Y,\star) \to (Z,\star)$, it holds

$$\check{\pi}_k^*(G^*F^*) = \check{\pi}_k^*(G^*)\check{\pi}_k^*(F^*).$$

Therefore, for every $k \in \mathbb{N}$, we have defined a functor

$$\check{\pi}_k^* : Sh_\star^* \to Grp$$

(for $k \geqslant 2$, instead of *Grp* we may put the category of abelian groups *Ab*).

Let $Set_\star$ be the category of pointed sets. Let us view $\check{\pi}_0^*(X,\star)$ as a pointed set having O^* as the base point. Now, the definition of the functors $\check{\pi}_k^*$ extends to the case $k = 0$ by taking $\check{\pi}_0^* : Sh_\star^* \to Set_\star$ to be a functor which associates with every pointed space $(X,\star)$ a pointed set $\check{\pi}_0^*(X,\star)$ and with every coarse shape morphism $F^* : (X,\star) \to (Y,\star)$ a base point preserving function $\check{\pi}_0^*(F^*) : \check{\pi}_0^*(X,\star) \to \check{\pi}_0^*(Y,\star)$, given by the rule in (2).

For every $k \in \mathbb{N}_0$, the functor $\check{\pi}_k^*$ associates with every pointed space the group (for $k = 0$ pointed set) $\check{\pi}_k^*(X,\star)$ which is called the k-th *coarse shape group* of X at the base point $\star$.

Remark 1. For every $k \in \mathbb{N}$ and every pointed pair of spaces $(X, X_0, \star)$ one also defines the k-th *relative coarse shape group* of a pair (X, X_0) at a base point $\star$, denoted by $\check{\pi}_k^*(X, X_0, \star)$, whose underlying set consists of all coarse shape morphisms $A^*: (D^k, S^{k-1}, \star) \to (X, X_0, \star)$ in the category $Sh_\star^*$ (D^k denotes the standard k-dimensional disk having S^{k-1} for its boundary). A group operation $+$ is given by formula (1), where coarse shape morphisms A^* and B^* are represented by morphisms $\boldsymbol{a}^* = [(a_\lambda^n)]$ and $\boldsymbol{b}^* = [(b_\lambda^n)] : (D^k, S^{k-1}, \star) \to (\boldsymbol{X}, \boldsymbol{X}_0, \star)$ in pro*-$HPol_\star^2$, respectively, and

$$\boldsymbol{p} : (X, X_0, \star) \to (\boldsymbol{X}, \boldsymbol{X}_0, \star) = \big((X_\lambda, X_{0\lambda}, \star), p_{\lambda\lambda'}, \Lambda\big)$$

is an $HPol_\star^2$-expansion of a pointed pair $(X, X_0, \star)$. Since elements of the k-th relative homotopy group $\pi_k(X_\lambda, X_{0\lambda}, \star)$ can be considered as H-maps $(D^k, S^{k-1}, \star) \to (X_\lambda, X_{0\lambda}, \star)$, the sum $a_\lambda^n + b_\lambda^n$ in (1) denotes the H-map which is a sum in the group $\pi_k(X_\lambda, X_{0\lambda}, \star)$. For every $k \geqslant 2$, $\check{\pi}_k^*(X, X_0, \star)$ is a group (for $k \geqslant 3$ an abelian group) and $\check{\pi}_1^*(X, X_0, \star)$ is a pointed set. A neutral element (base point) in $\check{\pi}_k^*(X, X_0, \star)$ is a coarse shape morphism $O^* = \langle \boldsymbol{o}^* \rangle$, $\boldsymbol{o}^* = [(o_\lambda^n)] : (D^k, S^{k-1}, \star) \to (\boldsymbol{X}, \boldsymbol{X}_0, \star)$, where $o_\lambda^n : (D^k, S^{k-1}, \star) \to (X_\lambda, X_{0\lambda}, \star)$ denotes the homotopy class of the map sending the disk D^k into X_0. In the relative case one also defines functors $\check{\pi}_k^* : Sh_\star^* \to Grp$, for $k \geqslant 2$, and $\check{\pi}_1^* : Sh_\star^* \to Set_\star$ which associate with every pointed pair $(X, X_0, \star)$ the group (for $k = 1$ pointed set) $\check{\pi}_k^*(X, X_0, \star)$ and associate with every coarse shape morphism $F^* : (X, X_0, \star) \to (Y, Y_0, \star)$ a homomorphism (for $k = 1$ a base point preserving function) $\check{\pi}_k^*(F^*) : \check{\pi}_k^*(X, X_0, \star) \to \check{\pi}_k^*(Y, Y_0, \star)$ given by (2).

3. Comparing the shape and the coarse shape groups

Let us recall that the functor pro-π_k, defined on the category $Sh_\star$, assigns to a pointed space $(X, \star)$ the inverse system pro-$\pi_k(X, \star) = (\pi_k(X_\lambda, \star), \pi_k(p_{\lambda\lambda'}), \Lambda)$ of groups (or pointed sets for $k = 0$), where $\boldsymbol{p} : (X, \star) \to ((X_\lambda, \star), p_{\lambda\lambda'}, \Lambda)$ is a fixed $HPol_\star$-expansion of $(X, \star)$. The functor $\check{\pi}_k : Sh_\star \to Grp$ ($\check{\pi}_0 : Sh_\star \to Set_\star$) associates with every pointed space $(X, \star)$ the group (pointed set) $\check{\pi}_k(X, \star)$, called the shape group (also fundamental group). It is usually defined by $\check{\pi}_k(X, \star) = \varprojlim(\text{pro-}\pi_k(X, \star))$ (see [5]). Although this approach to shape groups, because of its simplicity, is the standard one, here we use an alternative approach. According to that approach the underlying set of $\check{\pi}_k(X, \star)$ consists of all shape morphisms $A : (S^k, \star) \to (X, \star)$ and the group operation (for $k \geqslant 1$) is given by the formula

$$A + B = \langle \boldsymbol{a} \rangle + \langle \boldsymbol{b} \rangle = \langle \boldsymbol{a} + \boldsymbol{b} \rangle = \big\langle \big[(a_\lambda)\big] + \big[(b_\lambda)\big] \big\rangle = \big\langle \big[(a_\lambda + b_\lambda)\big] \big\rangle, \tag{3}$$

where shape morphisms A and B are represented by morphisms $\boldsymbol{a} = [(a_\lambda)]$ and $\boldsymbol{b} = [(b_\lambda)] : (S^k, \star) \to (\boldsymbol{X}, \star)$ in pro-$HPol_\star$, respectively, and $\boldsymbol{p} : (X, \star) \to (\boldsymbol{X}, \star) = ((X_\lambda, \star), p_{\lambda\lambda'}, \Lambda)$ is an $HPol_\star$-expansion of the pointed space $(X, \star)$. An H-map $a_\lambda + b_\lambda$ is the sum in $\pi_k(X_\lambda, \star)$. To every shape morphism $F : (X, \star) \to (Y, \star)$, $\check{\pi}_k$ assigns a homomorphism (a base point preserving function, for $k = 0$)

$$\check{\pi}_k(F) : \check{\pi}_k(X, \star) \to \check{\pi}_k(Y, \star),$$

by the following rule

$$\check{\pi}_k(F)(A) = FA, \tag{4}$$

for every $A \in \check{\pi}_k(X, \star)$. K. Borsuk has introduced this concept of shape (fundamental) groups for metric compacta, using his method of fundamental sequences (see [1]). The mentioned approaches to the shape groups are equivalent (see [6]) and they can be extended to all pointed spaces.

As we have already mentioned in the Introduction, the faithful functor $J : Sh_\star \to Sh_\star^*$ allows us to view the pointed shape category $Sh_\star$ as a subcategory of $Sh_\star^*$. It keeps the objects fixed, and to each shape morphism $F : (X, \star) \to (Y, \star)$, represented by $\boldsymbol{f} = [(f, f_\mu)] : (\boldsymbol{X}, \star) \to (\boldsymbol{Y}, \star)$, it assigns a coarse shape morphism $J(F) : (X, \star) \to (Y, \star)$, represented by $\boldsymbol{f}^* = [(f, f_\mu^n)] : (\boldsymbol{X}, \star) \to (\boldsymbol{Y}, \star)$, $f_\mu^n = f_\mu$, for every $\mu \in M$, $n \in \mathbb{N}$. Notice, that for every $A, B \in \check{\pi}_k(X, \star)$, $k \in \mathbb{N}$, $A = \langle [(a_\lambda)] \rangle$ and $B = \langle [(b_\lambda)] \rangle$, by (3), holds

$$J(A + B) = J\big\langle \big[(a_\lambda + b_\lambda)\big] \big\rangle = J\big\langle \big[(c_\lambda)\big] \big\rangle = \big\langle \big[(c_\lambda^n)\big] \big\rangle,$$

$c_\lambda^n = c_\lambda = a_\lambda + b_\lambda$, for every $\lambda \in \Lambda$, $n \in \mathbb{N}$. On the other side, by (1) one obtains

$$J(A) + J(B) = \big\langle \big[\big(a_\lambda^n\big)\big] \big\rangle + \big\langle \big[\big(b_\lambda^n\big)\big] \big\rangle = \big\langle \big[\big(a_\lambda^n + b_\lambda^n\big)\big] \big\rangle.$$

Since, $a_\lambda^n = a_\lambda$ and $b_\lambda^n = b_\lambda$, it follows that $c_\lambda^n = a_\lambda^n + b_\lambda^n$, for every $\lambda \in \Lambda$, $n \in \mathbb{N}$. Hence, $J(A + B) = J(A) + J(B)$. Therefore, a faithful functor J restricted to the group $\check{\pi}_k(X, \star)$ is an injective homomorphism (or injective base preserving function, for $k = 0$). That means that

$$j \equiv J|_{\check{\pi}_k(X, \star)} : \check{\pi}_k(X, \star) \to \check{\pi}_k^*(X, \star)$$

is an embedding and $\check{\pi}_k(X, \star)$ can be considered as a subgroup (subset) of $\check{\pi}_k^*(X, \star)$. Furthermore, for every shape morphism $F : (X, \star) \to (Y, \star)$, by (2) and (4) and by the functoriality of J, for every $A \in \check{\pi}_k(X, \star)$ and $k \in \mathbb{N}_0$, it follows

$$\begin{aligned}\big(\check{\pi}_k^*\big(J(F)\big)\circ j\big)(A) &= \check{\pi}_k^*\big(J(F)\big)\big(j(A)\big) = J(F)j(A) = J(F)J(A) = J(FA)\\ &= J\big(\check{\pi}_k(F)(A)\big) = \big(j\circ\check{\pi}_k(F)\big)(A).\end{aligned}$$

Therefore, we have proved the following theorem:

Theorem 2. *For every pointed space $(X,\star)$ and every $k\in\mathbb{N}_0$, the shape group $\check{\pi}_k(X,\star)$ can be considered a subgroup (subset) of the coarse shape group $\check{\pi}_k^*(X,\star)$, i.e. there exists an embedding $j:\check{\pi}_k(X,\star)\to\check{\pi}_k^*(X,\star)$. Moreover, for every shape morphism $F:(X,\star)\to(Y,\star)$ the following diagram commutes in the category Grp ($Set_\star$)*

$$\begin{tikzcd}
\check{\pi}_k(X,\star) \arrow[r, hook, "j"] \arrow[d, "\check{\pi}_k(F)"'] & \check{\pi}_k^*(X,\star) \arrow[d, "\check{\pi}_k^*(J(F))"] \\
\check{\pi}_k(Y,\star) \arrow[r, hook, "j"] & \check{\pi}_k^*(Y,\star).
\end{tikzcd}$$

For every $k\in\mathbb{N}_0$, with every pointed space $(X,\star)$ we can associate the homotopy group $\pi_k(X,\star)$, which is a homotopy invariant, the shape group $\check{\pi}_k(X,\star)$ which is both a homotopy and a shape invariant, and finally we can associate the coarse shape group which is not only a coarse shape invariant, but is also a shape and a homotopy invariant. Therefore, it could be useful to compute explicitly coarse shape groups of some particular spaces. We know that the homotopy groups and shape groups coincide on the class of polyhedra, but on that class these groups differ from the coarse shape groups, as we can see in the following example.

Example 1. Let $(P,\star)$ be a pointed polyhedron (or a pointed space having homotopy type of some pointed polyhedron) and $k\in\mathbb{N}$. Every coarse shape morphism $A^*:(S^k,\star)\to(P,\star)$ is represented by some morphism $\boldsymbol{a}^*=[(a^n)]:(S^k,\star)\to(P,\star)$ in pro*-*HPol*$_\star$. Notice that an S^*-morphism representing $\boldsymbol{a}^*$ is nothing else but a sequence of H-maps $a^n:(S^k,\star)\to(P,\star)$. Therefore, we may view an S^*-morphism (a^n) as a sequence in $\pi_k(P,\star)$, i.e. as an element a of the group $\prod_{n\in\mathbb{N}}\pi_k(P,\star)$. Notice that an S^*-morphism $(a^n+a'^n)$ corresponds to $a+a'$ in $\prod_{n\in\mathbb{N}}\pi_k(P,\star)$. Further, notice that $(a^n)\sim(a'^n):(S^k,\star)\to(P,\star)$ is equivalent to $a^n=a'^n$, for almost all $n\in\mathbb{N}$. Hence

$$\big(a^n\big)\sim\big(a'^n\big)\quad\Longleftrightarrow\quad a^n-a'^n=o,\quad\text{for almost all } n\in\mathbb{N}\quad\Longleftrightarrow\quad a-a'\in\bigoplus_{n\in\mathbb{N}}\pi_k(P,\star).$$

Since the direct sum $\bigoplus_{n\in\mathbb{N}}\pi_k(P,\star)$ is a normal subgroup of the product $\prod_{n\in\mathbb{N}}\pi_k(P,\star)$, one infers that $\check{\pi}_k^*(P,\star)$ is the quotient group $\prod_{n\in\mathbb{N}}\pi_k(P,\star)/\bigoplus_{n\in\mathbb{N}}\pi_k(P,\star)$, i.e. one obtains the formula

$$\check{\pi}_k^*(P,\star)=\Big(\prod_{n\in\mathbb{N}}G_n\Big)\Big/\Big(\bigoplus_{n\in\mathbb{N}}G_n\Big),\quad G_n=\pi_k(P,\star),\ n\in\mathbb{N}.$$

Since $\check{\pi}_k(X,\star)$ is a subgroup of $\check{\pi}_k^*(X,\star)$ it is natural to anticipate that coarse shape groups contain more information on a pointed space then the shape groups do. Next example confirms that expectation.

Example 2. Let $(D,\star)$ be the dyadic solenoid, i.e., $(D,\star)=\lim((X_i,\star),p_{ii+1})$, where $X_i=S^1=\{z\in\mathbb{C}\mid |z|=1\}$ and $p_{ii+1}(z)=z^2$. It is well known that $\check{\pi}_1(D,\star)=0$. On the other hand, $\check{\pi}_1^*(D,\star)\neq 0$. Moreover, $\check{\pi}_1^*(D,\star)$ has uncountable many elements. Indeed, since $\pi_1(S^1)=\mathbb{Z}$ every sequence $\alpha=(\alpha_k)$ in $\mathbb{Z}$ can be considered a sequence of H-maps $\alpha_k:(S^1,\star)\to(S^1,\star)$. Therefore, every sequence (α_k) determines an S^*-morphism $(a_i^n):(S^1,\star)\to((X_i,\star),p_{ii+1})$ given by

$$a_i^n:\big(S^1,\star\big)\to\big(S^1,\star\big),\quad a_i^n=\begin{cases}p_{in}\alpha_n, & i\leqslant n,\\ o, & i>n.\end{cases}$$

Obviously, $(a_i^n)\sim(a_i'^n)$ is equivalent to $\alpha_k=\alpha'_k$, for almost all $k\in\mathbb{N}$. Since there are uncountably many sequences in $\mathbb{Z}$, and each sequence α in $\mathbb{Z}$ determines a coarse shape morphism $A^*=\langle[(a_i^n)]\rangle:(S^1,\star)\to(D,\star)$ and, besides α, there are countably many sequences determining the same coarse shape morphism A^*, because only a sequence having all but finitely many terms equal to 0 determines the morphism O^*.

4. Relation with the *n*-shape connectedness

Recall that a pointed space $(X,\star)$ is said to be n-shape connected if pro-$\pi_k(X,\star)=\mathbf{0}$, for every $k\in\mathbb{N}_0$, $k\leqslant n$. Here $\mathbf{0}$ denotes the zero object in the category pro-*Grp* (or pro-*Set*$_\star$, for $k=0$), i.e., an inverse system of groups (pointed sets) isomorphic to the rudimentary inverse system (0). Since pro-$\pi_k(X,\star)$ is a pro-group and $\check{\pi}_k^*(X,\star)$ is a group, we are not able to compare them directly. However, the main goal of this section is to determine whether $\check{\pi}_k^*(X,\star)=0$ implies pro-$\pi_k(X,\star)=\mathbf{0}$ and conversely.

Theorem 3. *Let $(X, \star)$ be a pointed space and $k \in \mathbb{N}_0$. If* pro-$\pi_k(X, \star) = 0$*, then* $\check{\pi}_k^*(X, \star) = 0$.

Proof. Assume on the contrary that $\check{\pi}_k^*(X, \star) \neq 0$ is not a trivial group. Therefore, there exists a coarse shape morphism $A^* = \langle \boldsymbol{a}^* \rangle : (S^k, \star) \to (X, \star)$ such that $A^* \neq O^*$. If A^* is represented by $\boldsymbol{a}^* = [(a_\lambda^n)] : (S^k, \star) \to (\boldsymbol{X}, \star) = ((X_\lambda, \star), p_{\lambda\lambda'}, \Lambda)$, then $\boldsymbol{a}^* \neq \boldsymbol{o}^* = [(o_\lambda^n)]$, $o_\lambda^n = o$, and consequently $(a_\lambda^n) \nsim (o_\lambda^n)$. It follows that there exists a $\lambda \in \Lambda$ such that, for every $n \in \mathbb{N}$, there exists $n' \geqslant n$ such that $a_\lambda^{n'} \neq o$. Now, for every $\lambda' \geqslant \lambda$, there exists n' such that

$$p_{\lambda\lambda'} a_{\lambda'}^{n'} = a_\lambda^{n'} \neq o. \tag{5}$$

Indeed, for every $\lambda' \geqslant \lambda$, there exists an $n(\lambda') \in \mathbb{N}$ such that

$$p_{\lambda\lambda'} a_{\lambda'}^n = a_\lambda^n,$$

for every $n \geqslant n(\lambda')$, and there exists $n' \geqslant n(\lambda')$ such that $a_\lambda^{n'} \neq o$, which implies (5). Now, for every $\lambda' \geqslant \lambda$, by (5), one infers that for the homomorphism

$$\pi_k(p_{\lambda\lambda'}) : \pi_k(X_{\lambda'}, \star) \to \pi_k(X_\lambda, \star),$$

there exists an element $a_{\lambda'}^{n'} \in \pi_k(X_{\lambda'}, \star)$ such that

$$\pi_k(p_{\lambda\lambda'})\big(a_{\lambda'}^{n'}\big) \neq o,$$

which implies that, for every $\lambda' \geqslant \lambda$ the homomorphism $\pi_k(p_{\lambda\lambda'})$ is not trivial. By Theorem 2.2.7 in [5] (see also Theorem 4.4 and Corollary 4.6 in [3]), it follows that the inverse system $(\pi_k(X_\lambda, \star), \pi_k(p_{\lambda\lambda'}), \Lambda)$ is not the zero object in pro-*Grp* (pro-*Set*$_\star$), i.e., pro-$\pi_k(X, \star) \neq \mathbf{0}$, which contradicts the assumption of the theorem. □

Theorem 4. *Let $(X, \star)$ be a pointed compact metric space and $k \in \mathbb{N}_0$. If $\check{\pi}_k^*(X, \star) = 0$, then* pro-$\pi_k(X, \star) = 0$.

Proof. Let $\boldsymbol{p} : (X, \star) \to (\boldsymbol{X}, \star) = ((X_i, \star), p_{ii+1}, \mathbb{N})$ be a sequential *HPol*$_\star$-expansion of $(X, \star)$. Assume, on the contrary, that pro-$\pi_k(X, \star) \neq \mathbf{0}$. Then, by Theorem 2.2.7 in [5], it follows that there exists $i_0 \in \mathbb{N}$ such that, for every $i \geqslant i_0$, $\pi_k(p_{i_0 i})$ is not a trivial homomorphism. There is no loss of generality in assuming that $i_0 = 1$ (if we delete in $(\boldsymbol{X}, \star)$ all terms $(X_i, \star)$, $i < i_0$, one obtains a new inverse sequence which is an *HPol*$_\star$-expansion of $(X, \star)$ having $(X_{i_0}, \star)$ for its first term). Hence, for every $i \geqslant 1$, there exists an element $a_i \in \pi_k(X_i, \star)$ such that

$$\pi_k(p_{1i})(a_i) \neq o,$$

which implies that the H-map $a_i : (S^k, \star) \to (X_i, \star)$ has the following property

$$o \neq p_{1i} a_i : \big(S^k, \star\big) \to (X_1, \star). \tag{6}$$

We will now define H-maps $b_i^n : (S^k, \star) \to (X_i, \star)$, for all $i, n \in \mathbb{N}$, by putting

$$b_i^n = \begin{cases} p_{in} a_n, & i \leqslant n, \\ o, & i > n. \end{cases}$$

Obviously, for every pair $i \leqslant i'$ and for every $n \geqslant i'$, it holds

$$b_i^n = p_{in} a_n = p_{ii'} p_{i'n} a_n = p_{ii'} b_{i'}^n.$$

Hence, $(b_i^n) : (S^k, \star) \to (\boldsymbol{X}, \star)$ is an S^*-morphism. Notice that, for every $i \in \mathbb{N}$ and for every $n \geqslant i$, $b_i^n \neq o$. Otherwise, $o = b_i^n = p_{in} a_n$ would imply $o = p_{1i} p_{in} a_n = p_{1n} a_n$, which is impossible because of (6). Therefore $(b_i^n) \nsim (o_i^n)$, $o_i^n = o$, for all $i, n \in \mathbb{N}$. Now, it follows

$$\big[\big(b_i^n\big)\big] = \boldsymbol{b}^* \neq \boldsymbol{o}^* = \big[\big(o_i^n\big)\big],$$

which implies $O^* \neq B^*$, $B^* = \langle \boldsymbol{b}^* \rangle : (S^k, \star) \to (X, \star)$. This is a contradiction to the assumption $\check{\pi}_k^*(X, \star) = 0$. □

Corollary 1. *Let $(X, \star)$ be a pointed compact metric space. For every $k \in \mathbb{N}_0$, $\check{\pi}_k^*(X, \star) = 0$ if and only if* pro-$\pi_k(X, \star) = 0$. *Particularly, $(X, \star)$ is n-shape connected if and only if $\check{\pi}_k^*(X, \star) = 0$, for every $k \leqslant n$.*

Remark 2. Using relative coarse shape groups instead of coarse shape groups and performing obvious changes in the proofs of Theorems 2, 3 and 4, one can easily see that analogous statements hold also in the relative case.

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